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# Coherent $H^\infty$ control for Markovian jump linear quantum systems

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**Abstract:** The purpose of this paper is to design a coherent feedback controller for a Markovian jump linear quantum system suffering from a fault signal. The control objective is to bound the effect of the disturbance input on the output for the time-varying quantum system. We prove the relation between the  $H^\infty$  control problem, the dissipation properties, and the solutions of Riccati differential equations, by which the  $H^\infty$  controller of the Markovian jump linear quantum system is given by the solutions of Linear Matrix Inequalities (LMIs).

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**Keywords:** Coherent quantum feedback,  $H^\infty$  control, fault-tolerant control, Markovian jump linear quantum systems.

## 1. INTRODUCTION

Controlling quantum phenomena has become a critical task in quantum technology, quantum optics and chemical physics Altafini and Ticozzi (2012); Dong and Petersen (2010); Wiseman and Milburn (2009); Guo et al. (2019); Shu et al. (2020); Wu et al. (2019); Li and Khaneja (2009). Quantum control systems may suffer from different faults and uncertainties in practical applications (Xiang et al., 2016, 2017; Dong et al., 2019; Yamamoto, 2006). For example, the fluctuations of the classical lasers in quantum optics or fault operations in the generators of quantum resources may introduce fault signals, leading to a deterioration of the performance of the system or causing the system to become unstable (Wang and Dong, 2016). Since a quantum system has unique features such as measurement back action and noncommutative observables

(Dong and Petersen, 2010; Liu et al., 2019), some classical fault-tolerant control strategies cannot be applied directly (Wang and Dong, 2016). This paper aims to develop fault-tolerant feedback control theory for a class of linear quantum systems with fault signals. Feedback control, including measurement-based feedback control and coherent feedback control, may have good robustness compared with open loop control (Liu et al., 2016; Zhang and James, 2010). We consider coherent feedback control in this paper, which can avoid time-delay and hardware mismatch issues since the controller itself is a quantum system (James et al., 2008; Nurdin et al., 2009; Maalouf and Petersen, 2010).

$H^\infty$  control is a well known robust control method used in many classical systems. It has also been widely used in quantum cases to bound the influence of the disturbance input signals on the output (James et al., 2008; Maalouf and Petersen, 2010; Wang et al., 2017). Based on the quantum version of standard dissipation properties, James et al. (2008) represented the  $H^\infty$  control problem for quantum systems and a solution involving two Riccati equations. The controller can then be constructed from the Riccati equation solutions, and is implemented by a fully classical

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system, a purely quantum system or a mixture of quantum and classical elements. While these results only considered the cases of time-invariant quantum systems, in practical applications, time-varying linear quantum systems may be often encountered. A dynamic game approach has been proposed to solve the time-varying  $H^\infty$  control problem in linear quantum systems (Maalouf and Petersen, 2012), where the designed controller is a classical system. This paper aims to solve the time-varying  $H^\infty$  coherent feedback control problem for a linear quantum system suffering from a fault signal. The dissipation properties of the time-varying quantum systems are presented, by which the  $H^\infty$  control problem can be transformed to finding the solutions of Riccati differential equations and a group of LMIs. In many practical applications, some faults (for example the fluctuations of the laser in quantum optics) can be modelled as a Markov chain on a probability space. Therefore the whole system becomes a Markovian jump linear quantum system. The  $H^\infty$  control performance of a Markovian linear jump quantum system can be related to a corresponding classical system. Hence, we can refer to the  $H^\infty$  control design method in classical cases to design a controller for a Markovian jump linear quantum system.

The rest of this paper is organized as follows. Section 2 presents the system model and the problem formulation. In Section 3, a theorem is obtained to illustrate the equivalence between the dissipation properties, the  $H^\infty$  control problem, and a part of Riccati differential equations. The controller for a Markovian jump linear quantum system is designed in Section 4, where the controller is solved by a group of LMIs. The conclusion is given in Section 5.

## 2. SYSTEM MODEL AND PROBLEM FORMULATION

Linear quantum systems are commonly met in quantum optics, and can be described by the following differential linear equations

$$\begin{aligned} dx(t) &= Ax(t)dt + Bd\omega(t); x(0) = x_0, \\ dz(t) &= Cx(t)dt + Dd\omega(t), \end{aligned} \quad (1)$$

where  $A, B, C, D$  are real matrices with appropriate dimensions  $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_\omega}, C \in \mathbb{R}^{n_y \times n}, D \in \mathbb{R}^{n_y \times n_\omega}$ , and  $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$  is a vector of self-adjoint possibly noncommutative system variables.  $z(t) = [z_1(t), z_2(t), \dots, z_{n_y}(t)]^T$  represents the output variables. The initial variables of the quantum systems satisfy the commutation relations

$$[x_i(0), x_j(0)] = 2i\Theta_{jk}, j, k = 1, \dots, n. \quad (2)$$

Here the commutator is defined by  $[A, B] = AB - BA$ .  $\Theta = [\Theta_{jk}]$  is a real anti-symmetric matrix, and is in one of the following forms:

- Canonical if  $\Theta = \text{diag}(J, J, \dots, J)$ ,
- Degenerate canonical if  $\Theta = \text{diag}(0_{n' \times n'}, J, \dots, J)$ ,

with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

A degenerate  $\Theta$  means the system contains classical information. Here,  $\omega$  represents the disturbance input, and is assumed to have the form

$$d\omega(t) = \beta_\omega(t)dt + d\tilde{\omega}(t), \quad (3)$$

where  $\beta_\omega(t)$  is a self adjoint process and  $\tilde{\omega}(t)$  is the noise part. The quantum noise satisfies Ito table condition  $d\tilde{\omega}(t)d\tilde{\omega}^T(t) = F_{\tilde{\omega}}dt$  with a non-negative matrix  $F_{\tilde{\omega}}$  (Belavkin, 1992). We write

$$S_{\tilde{\omega}} = \frac{1}{2}(F_{\tilde{\omega}} + F_{\tilde{\omega}}^T), \quad (4)$$

and

$$T_{\tilde{\omega}} = \frac{1}{2}(F_{\tilde{\omega}} - F_{\tilde{\omega}}^T), \quad (5)$$

where  $T_{\tilde{\omega}}$  satisfies the following equation

$$[d\tilde{\omega}(t), d\tilde{\omega}^T(t)] = d\tilde{\omega}(t)d\tilde{\omega}^T(t) - (d\tilde{\omega}(t)d\tilde{\omega}^T(t))^T = 2T_{\tilde{\omega}}dt. \quad (6)$$

The above linear differential equation can describe many practical quantum systems, for example, an open quantum harmonic oscillator with a quadratic Hamiltonian and a coupling operator in (James et al., 2008). In some quantum optical experiments, e.g., an Optical Parametric Amplifier (OPO) used to generate the squeezed light, the pumping field is usually treated as a classical laser. If the laser device is subject to a fault process, a time-varying Hamiltonian will be introduced to the linear differential equations, which will lead to a time-varying linear quantum system. In this case, the system Hamiltonian can be described as  $H(F(t))$ , where  $F(t)$  is the fault process (Gao et al., 2016), which introduces a time-varying matrix  $A(F(t))$  in the linear differential equations (1). In the following, a time-varying open quantum harmonic oscillator is first defined, which will be used to illustrate the physical realisation of the quantum systems.

*Definition 1.* The system (1) with time-varying  $A(F(t))$  (also with  $\beta_\omega = 0$ ) is said to be an open quantum harmonic oscillator if  $\Theta$  is canonical and there exist a quadratic Hamiltonian  $H = \frac{1}{2}x(0)^T R(t)x(0)$ , with a real and symmetric Hamiltonian matrix  $R$  of dimension  $n \times n$ , and a coupling operator  $L = \Lambda x(0)$ , with complex-valued coupling matrix  $\Lambda$  of dimension  $n_\omega \times n$ , such that

$$\begin{aligned} x_k(t) &= U(t)^* x_k(0) U(t), k = 1, \dots, n, \\ z_l(t) &= U(t)^* \omega_l(0) U(t), l = 1, \dots, n_y, \end{aligned}$$

where  $\{U(t); t \geq 0\}$  is an adapted process of unitary operators satisfying the following Quantum Stochastic Differential Equations (QSDE) (James et al., 2008)

$$\begin{aligned} dU(t) &= (-iH(F(t))dt - \frac{1}{2}L^\dagger L dt + [-L^\dagger L^T] \Gamma d\omega(t))U(t), \\ U(0) &= I. \end{aligned}$$

In this case, the matrices  $A, B, C, D$  are given by

$$A = 2\Theta(R(F(t)) + \Im(\Lambda^\dagger \Lambda)), \quad (7)$$

$$B = 2i\Theta[-\Lambda^\dagger \Lambda^T] \Gamma, \quad (8)$$

$$C = P_{N_y}^T \begin{bmatrix} \sum_{N_y} 0_{N_y \times N_\omega} \\ 0_{N_y \times N_\omega} \sum_{N_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix}. \quad (9)$$

Here,  $N_\omega = \frac{n_\omega}{2}, N_y = \frac{n_y}{2}, \Gamma = P_{N_\omega} \text{diag}_{N_\omega}(M)$ , where

$$M = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix},$$

and  $\sum_{N_y} = [I_{N_y \times N_y} \ 0_{N_y \times (N_\omega - N_y)}]$ .  $P_k$  is the permutation matrix satisfying

$$P_k a = [a_1 \ a_3 \ \dots \ a_{2k-1} \ a_2 \ a_4 \ \dots \ a_{2k}]^T,$$

where  $a = [a_1 \ a_2 \ \dots \ a_{2k}]^T$ .

This results in a time-varying linear quantum system

$$\begin{aligned} dx(t) &= A(t)x(t)dt + Bd\omega(t); x(0) = x_0, \\ dz(t) &= Cx(t)dt + Dd\omega(t), \end{aligned} \quad (10)$$

where  $A(t) = 2\Theta(R(F(t)) + \Im(\Lambda^\dagger\Lambda))$ . Other system variables are defined to be the same as that in (1).

This paper aims to analyse the dissipation properties of the time-varying stochastic linear quantum systems described by (10), and apply the robust  $H^\infty$  control method to ensure that this quantum system is strictly bounded real with any given disturbance attenuation  $g$ .

### 3. DISSIPATION PROPERTIES

Dissipation properties state the relation between storage function and the supply functions in terms of system energy in classical systems (Willems, 1972). In this section, we consider the dissipation properties for the time-varying quantum systems (10).

We first define a storage function  $V(x(t)) = x(t)^T P(t)x(t)$ , where  $P(t)$  is a time-varying positive definite symmetric matrix. We consider an operator valued quadratic function

$$\gamma(x, \beta_\omega) = \begin{bmatrix} x^T & \beta_\omega^T \end{bmatrix}^T S \begin{bmatrix} x \\ \beta_\omega \end{bmatrix},$$

where  $S$  is a constant real symmetric matrix, called the supply rate function.

The following definition presents the dissipation inequality for a time-varying quantum system.

*Definition 2.* The system (10) is said to be dissipative with supply rate  $\gamma(x, \beta_\omega)$  if there exists a positive time-varying storage function  $V(x(t)) = x(t)^T P(t)x(t)$  and a constant  $\lambda > 0$  such that

$$\langle V(x(t)) \rangle + \int_0^t \langle \gamma(x(s), \beta_\omega(s)) \rangle ds \leq \langle V(x(0)) \rangle + \lambda t, \forall t > 0, \quad (11)$$

where  $\langle V(x(t)) \rangle$  represents the expectation of the operator  $V(x(t))$ .

The system (10) is said to be strictly dissipative if there exists a constant  $\epsilon > 0$  such that inequality (11) holds with supply rate  $S$  replaced by  $S + \epsilon I$ .

*Definition 3.* (James et al., 2008) The quantum system (10) is bounded real with disturbance attenuation  $g$  if the system is dissipative with

$$\begin{aligned} \gamma(x, \beta_\omega) &= \beta_z^T \beta_z - g^2 \beta_\omega^T \beta_\omega \\ &= \begin{bmatrix} x^T & \beta_\omega^T \end{bmatrix} \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - g^2 I \end{bmatrix} \begin{bmatrix} x \\ \beta_\omega \end{bmatrix}, \end{aligned} \quad (12)$$

where  $\beta_z(t) = Cx(t) + D\beta_\omega(t)$ . Also we say that the system (10) is strictly bounded real with disturbance attenuation  $g$  if the system is strictly dissipative with supply rate (12).

With these definitions, the following theorem states the relation between the dissipation properties, the Riccati differential equations, and the  $H^\infty$  control problem, which will be used to design a coherent controller.

*Theorem 1.* For the system (10), the following four statements are equivalent

- 1) The system (10) is strictly bounded real with disturbance attenuation  $g$ ;

- 2) There exists a positive definite matrix  $\tilde{P}(t)$  such that
 
$$\begin{aligned} &\dot{\tilde{P}}(t) + A(t)^T \tilde{P}(t) + \tilde{P}(t)A(t) + C^T C \\ &+ (C^T D + \tilde{P}(t)B)(g^2 I - D^T D)^{-1}(D^T C + B^T \tilde{P}(t)) \\ &< 0; \end{aligned}$$

- 3) The Riccati differential equation

$$\begin{aligned} &\dot{P}(t) + A(t)^T P(t) + P(t)A(t) + C^T C \\ &+ (C^T D + P(t)B)(g^2 I - D^T D)^{-1}(D^T C + B^T P(t)) \\ &= 0 \end{aligned}$$

has a stabilizing solution  $P(t) \geq 0$ ;

- 4) The homogeneous system  $\dot{x}(t) = A(t)x(t)$  is exponentially stable, and the operator mapping  $\omega$  to  $z$  satisfies  $\|T_{z\omega}\|_\infty < g$ .

**Proof.** The proof details can be found in (Liu et al., 2020).

### 4. $H^\infty$ CONTROL DESIGN FOR MARKOVIAN JUMP LINEAR SYSTEMS

In practice, it is possible that the system transits between a finite number of different faulty modes at random times. This makes it desirable to model the fault process as a continuous-time Markov chain  $\{F(t)\}_{t \geq 0}$  on a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$  (Gao et al., 2016), which results in a Markovian jump linear system (MJLSs). The fault process  $F(t)$  is a continuous-time, discrete-state Markovian process on the probability space, with state space defined as  $\mathbb{S} = \{e_1, e_2, \dots, e_N\}$  for an integer  $N$ . Hence the quantum system with this fault signal becomes a Markovian jump linear system. MJLSs have been widely studied in classical systems, since they are suitable models to describe a class of systems suffering from a random abrupt variations in their structures (Xiong et al., 2005). In this section, we consider MJLSs in quantum cases. We suppose the Markov process has a known transition rate matrix  $\Pi = (\pi_{jk}) \in \mathbb{R}^{N \times N}$ , where we have  $\pi_{jj} = -\sum_{j \neq k} \pi_{jk}$ , and  $\pi_{jk} \geq 0, j \neq k$ .

#### 4.1 Closed-loop Markovian jump linear systems

The system with a disturbance input and a control input is described as

$$\begin{aligned} dx(t) &= A(F(t))x(t)dt + B_1 d\omega(t) + B_2 du(t), \\ dz(t) &= C_1 x(t)dt + D_1 du(t), \\ dy(t) &= C_2 x(t)dt + D_2 d\omega(t). \end{aligned} \quad (13)$$

Here,  $A(F(t))$  takes finite values in  $(A_1, A_2, \dots, A_N)$  due to the fault process;  $y(t)$  is the measured output and  $z(t)$  represents the error output. The control signal satisfies

$$du(t) = \beta_u(t)dt + \tilde{u}(t), \quad (14)$$

where  $\beta_u(t)$  is an adapted process,  $\tilde{u}(t)$  is the quantum noise part of  $u(t)$  with the Ito matrix  $F_{\tilde{u}}$ .

Suppose the controller is described by the following dynamical equations

$$\begin{aligned} d\xi(t) &= A_{K_i} \xi(t)dt + B_{K_i} dy(t) + B_{\nu_i} d\nu_K(t), \\ du(t) &= C_{K_i} \xi(t)dt + D_{\nu_i} d\nu_K(t), \end{aligned} \quad (15)$$

where  $\xi(t) = [\xi_1(t) \ \xi_2(t) \ \dots \ \xi_{n_k}(t)]^T$  is a vector of self-adjoint controller variables. The noise  $\nu_K$  is a vector of

noncommutative Wiener processes satisfying the Ito table consider with canonical Hermitian Ito matrix  $F_{\nu_K}$ . The designed controller also jumps between different modes with  $\{(A_{K1}, B_{K1}, C_{K1}), \dots, (A_{KN}, B_{KN}, C_{KN})\}$ , and this switching is based on the modes of the plant. In this paper, we assume that the transit rate matrix of the Markovian plant is precisely known and the mode of the plant is accessible to the controller.

We obtain the closed-loop systems by identifying  $\beta_u(t) = C_K(t)\xi(t)$  as

$$d\eta(t) = \begin{bmatrix} A_i & B_2 C_{Ki} \\ B_{Ki} C_2 & A_{Ki} \end{bmatrix} \eta(t) dt + \begin{bmatrix} B_1 \\ B_{Ki} D_2 \end{bmatrix} d\omega(t) + \begin{bmatrix} B_2 D_{\nu i} \\ B_{\nu i} \end{bmatrix} d\nu_K(t), \quad (16)$$

$$dz(t) = [C_1 \ D_1 C_{Ki}] \eta(t) dt + D_1 D_{\nu i} d\nu_K(t),$$

$$\text{with } \eta(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}.$$

#### 4.2 Control objective

The control objective here is to design a controller (15) such that the closed-loop system (16) is strictly bounded real with a given disturbance attenuation  $g$ , which means

$$\begin{aligned} & \langle \eta^T(t) P(t) \eta(t) \rangle + \int_0^t \langle \beta_z^T(s) \beta_z(s) - g^2 \beta_\omega^T(s) \beta_\omega(s) \\ & + \epsilon \eta^T(s) \eta(s) + \epsilon \beta_\omega^T(s) \beta_\omega(s) \rangle ds \\ & \leq \langle \eta^T(0) P_0 \eta(0) \rangle + \lambda t, \forall t > 0. \end{aligned} \quad (17)$$

In this case, we understand that the control objective is to bound the effect of the energy of disturbance input  $\beta_\omega(t)$  on the energy of the error output  $\beta_z(t)$ .

Before we apply the  $H^\infty$  method to design a controller for the quantum system (13), the equivalence of the control performance between the quantum system and its corresponding classical system can be considered.

We consider a corresponding classical system as follows:

$$\begin{aligned} dx_c(t) &= A(F(t))x_c(t)dt + B_1 d\omega_c(t) + B_2 du_c(t), \\ dz_c(t) &= C_1 x_c(t)dt + D_1 du_c(t), \\ dy_c(t) &= C_2 x_c(t)dt + D_2 d\omega_c(t), \end{aligned} \quad (18)$$

where  $d\omega_c(t) = \beta_\omega(t)dt + S_\omega^{1/2}d\tilde{\omega}(t)$ , and  $x_c(0)$  is a Gaussian random vector with mean  $\hat{x}_{c0}$  and covariance matrix  $Y_{c0}$ . Here  $S_\omega$  and  $S_u$  are defined as in (4).

We assume the controller of this classical system is described as follows

$$\begin{aligned} d\xi_c(t) &= A_{Ki}\xi_c(t)dt + B_{Ki}dy_c(t) + B_{\nu i}S_{\nu K}^{1/2}d\nu_K(t), \\ du_c(t) &= C_{Ki}\xi_c(t)dt + D_{\nu i}S_{\nu K}^{1/2}d\nu_K(t). \end{aligned} \quad (19)$$

In this case, the closed loop system is

$$\begin{aligned} d\eta_c(t) &= \begin{bmatrix} A_i & B_2 C_{Ki} \\ B_{Ki} C_2 & A_{Ki} \end{bmatrix} \eta_c(t) dt + \begin{bmatrix} B_1 \\ B_{Ki} D_2 \end{bmatrix} d\omega_c(t) \\ &+ \begin{bmatrix} B_2 D_{\nu i} \\ B_{\nu i} \end{bmatrix} S_{\nu K}^{1/2} d\nu_K(t), \\ dz_c(t) &= [C_1 \ D_1 C_{Ki}] \eta_c(t) dt + D_1 D_{\nu i} S_{\nu K}^{1/2} d\nu_K(t), \end{aligned} \quad (20)$$

where

$$\eta_c(t) = \begin{bmatrix} x_c(t) \\ \xi_c(t) \end{bmatrix}.$$

Also, let

$$\eta_c(0) = \eta_{c0} = \begin{bmatrix} x_c(0) \\ \xi_c(0) \end{bmatrix}.$$

Substituting  $d\omega(t) = \beta_\omega(t)dt + d\tilde{\omega}(t)$  and  $du(t) = \beta_u(t)dt + d\tilde{u}(t)$  into (16), we have

$$\begin{aligned} d\eta(t) &= \tilde{A}_i \eta(t) dt + \tilde{B}_{1i} \beta_\omega(t) dt + \tilde{B}_{1i} d\tilde{\omega}(t) + \tilde{B}_{2i} d\nu_K(t), \\ dz(t) &= \tilde{C}_i \eta(t) dt + \tilde{D}_i d\nu_K(t). \end{aligned} \quad (21)$$

Here,

$$\begin{aligned} \tilde{A}_i &= \begin{bmatrix} A_i & B_2 C_{Ki} \\ B_{Ki} C_2 & A_{Ki} \end{bmatrix}, \\ \tilde{B}_{1i} &= \begin{bmatrix} B_1 \\ B_{Ki} D_2 \end{bmatrix}, \tilde{B}_{2i} = \begin{bmatrix} B_2 D_{\nu i} \\ B_{\nu i} \end{bmatrix}, \\ \tilde{C}_i &= [C_1 \ D_1 C_{Ki}], \\ \tilde{D}_i &= D_1 D_{\nu i}. \end{aligned}$$

Similarly, we write (20) as

$$\begin{aligned} d\eta_c(t) &= \tilde{A}_i \eta_c(t) dt + \tilde{B}_{1i} \beta_\omega(t) dt + \tilde{B}_{1i} S_\omega^{1/2} d\tilde{\omega}(t) + \\ &\tilde{B}_{2i} S_{\nu K}^{1/2} d\nu_K(t), \\ dz_c(t) &= \tilde{C}_i \eta_c(t) dt + \tilde{D}_i S_{\nu K}^{1/2} d\nu_K(t). \end{aligned} \quad (22)$$

The control objective here is to make the system strictly bounded real with disturbance attenuation  $g$ , which means

$$\langle V(\eta(t)) \rangle + \int_0^t \langle \gamma(\eta(s), \beta(s)) \rangle ds \leq \langle V(\eta(0)) \rangle + \lambda t, t \geq 0. \quad (23)$$

For the quantum system, we have

$$\langle V(\eta(t)) \rangle = \langle \eta^T(t) P(t) \eta(t) \rangle,$$

with a positive-definite matrix  $P(t)$  and

$$\begin{aligned} \gamma(\eta(t), \beta_\omega(t)) &= \beta_z(t)^T \beta_z(t) - g^2 \beta_\omega^T(t) \beta_\omega(t) + \epsilon \eta^T(t) \eta(t) + \epsilon \beta_\omega^T(t) \beta_\omega(t) \\ &= \eta^T(t) [\tilde{C}_i^T \tilde{C}_i + \epsilon I] \eta(t) - (g^2 - \epsilon) I \beta_\omega^T(t) \beta_\omega(t). \end{aligned} \quad (24)$$

Substituting (24) into (23) gives

$$\begin{aligned} & \langle \eta^T(t) P(t) \eta(t) \rangle + \int_0^t \langle \eta^T(s) [\tilde{C}_i^T \tilde{C}_i + \epsilon I] \eta(s) \rangle ds \\ & - \int_0^t \langle (g^2 - \epsilon) \beta_\omega^T(s) \beta_\omega(s) \rangle ds \\ & \leq \langle \eta_0^T P_0 \eta_0 \rangle + \lambda t. \end{aligned} \quad (25)$$

Define  $Q(t) = \frac{1}{2} \langle \eta(t) \eta^T(t) + (\eta(t) \eta^T(t))^T \rangle$ . We have  $\langle \eta^T(t) P(t) \eta(t) \rangle = \text{Tr} \{ \tilde{P}(t) Q(t) \}$ , where

$$\tilde{P}(t) = \begin{bmatrix} P(t) & 0 \\ 0 & 0 \end{bmatrix},$$

$$\langle \eta^T(t) [\tilde{C}_i^T \tilde{C}_i + \epsilon I] \eta(t) \rangle = \text{Tr} \left\{ [\tilde{C}_i^T \tilde{C}_i + \epsilon I] Q(t) \right\}.$$

Hence, (25) becomes

$$\begin{aligned} & \text{Tr} \{ \tilde{P}(t) Q(t) \} + \int_0^t \text{Tr} \left\{ [\tilde{C}_i^T \tilde{C}_i + \epsilon I] Q(t) \right\} ds \\ & - (g^2 - \epsilon) \int_0^t \langle \beta_\omega^T(s) \beta_\omega(s) \rangle ds \\ & \leq \langle \eta^T(0) P_0 \eta(0) \rangle + \lambda t. \end{aligned} \quad (26)$$

Similarly, for the classical systems (18), if we may define  $\langle V(\eta_c(t)) \rangle = E(\eta_c^T(t)P(t)\eta_c(t))$ ,  $\gamma(\eta_c(t), \beta_\omega(t)) = \beta_{zc}^T(t)\beta_{zc}(t) - g^2\beta_\omega^T(t)\beta_\omega(t) + \epsilon\eta_c^T(t)\eta_c(t) + \epsilon\beta_\omega^T(t)\beta_\omega(t) = \eta_c^T(t) \left[ \tilde{C}_i^T \tilde{C}_i + \epsilon I \right] \eta_c(t) - (g^2 - \epsilon)\beta_\omega^T(t)\beta_\omega(t)$ , (27)

and  $Q'(t) = E(\eta_c(t)\eta_c^T(t))$ . Here  $E(\cdot)$  denotes the stochastic expectation. We have

$$E(\eta_c^T(t)P(t)\eta_c(t)) = \text{Tr} \left[ \tilde{P}(t)Q'(t) \right],$$

$$E \left\{ \eta_c^T(t) \left[ \tilde{C}_i^T \tilde{C}_i + \epsilon I \right] \eta_c(t) \right\} = \text{Tr} \left\{ \left[ \tilde{C}_i^T \tilde{C}_i + \epsilon I \right] Q'(t) \right\}.$$

The classical system (18) is strictly bounded real with  $g$  means

$$\begin{aligned} & \text{Tr} \left\{ \tilde{P}(t)Q'(t) \right\} + \int_0^t \text{Tr} \left\{ \left[ \tilde{C}_i^T \tilde{C}_i + \epsilon I \right] Q'(s) \right\} ds \\ & - (g^2 - \epsilon) \int_0^t \beta_\omega^T(s)\beta_\omega(s) ds \\ & \leq \langle \eta_c^T(0)P_0\eta_c(0) \rangle + \lambda t. \end{aligned} \quad (28)$$

We then calculate the differential of  $Q(t)$  and  $Q'(t)$

$$\begin{aligned} dQ(t) &= \tilde{A}(t)Q(t) + Q(t)\tilde{A}^T(t) \\ &+ \langle \eta(t) \rangle \beta_\omega^T(t) \tilde{B}_{1i}^T dt + \tilde{B}_{1i}\beta_\omega(t) \langle \eta^T(t) \rangle dt \\ &+ \tilde{B}_{1i}S_\omega(t) \tilde{B}_{1i}^T dt + \tilde{B}_{2i}S_{\nu\kappa}(t) \tilde{B}_{2i}^T dt, \\ dQ'(t) &= \tilde{A}(t)Q'(t) + Q'(t)\tilde{A}^T(t) \\ &+ E(\eta_c(t))\beta_\omega^T(t) \tilde{B}_{1i}^T dt + \tilde{B}_{1i}\beta_\omega(t)E(\eta_c^T(t))dt \\ &+ \tilde{B}_{1i}S_\omega(t) \tilde{B}_{1i}^T dt + \tilde{B}_{2i}S_{\nu\kappa}(t) \tilde{B}_{2i}^T dt. \end{aligned} \quad (29)$$

Also, we further calculate

$$\begin{aligned} \frac{d\langle \eta(t) \rangle}{dt} &= \tilde{A}(t)\langle \eta(t) \rangle + \tilde{B}_{1i}\beta_\omega(t), \\ \frac{E(\eta_c(t))}{dt} &= \tilde{A}(t)E(\eta_c(t)) + \tilde{B}_{1i}\beta_\omega(t). \end{aligned} \quad (30)$$

Note that if let the mean of the Gaussian state  $\langle \eta(0) \rangle = \tilde{\eta}_0 = \tilde{\eta}_{c0} = E(\eta_c(0))$ , we then have  $\langle \eta(t) \rangle \equiv E(\eta_c(t))$ . Moreover, we obtain  $Q(t) \equiv Q'(t)$ . This means that if the classical systems with controller in the form of (19) is strictly bounded real with disturbance attenuation  $g$ , the quantum system with the same control parameters in controller (15) is also strictly bounded real with  $g$ .

### 4.3 $H^\infty$ control design

The following proposition has been widely used in classical systems to design an  $H^\infty$  control law.

*Proposition 4.* (De Farias et al., 2000) For system (10), if there exists  $P = (P_1, \dots, P_N), P_i > 0$  satisfies

$$A_i^T P_i + P_i A_i + \sum_{j=1}^N \pi_{ij} P_j + g^{-2} P_i B B^T P_i + C^T C < 0, \quad (31)$$

for  $i = 1, \dots, N$ , then  $\|T_{\omega z}\|_\infty < g$ .

Here, the norm  $\|T_{\omega z}\|_\infty$  is the  $H^\infty$ -norm for the system from disturbance input  $\omega(t)$  to the error output  $z(t)$ .

We apply the above proposition to the closed-loop quantum system (16), and have the following conclusion.

*Theorem 5.* If there exists a controller of the form (15) such that the closed-loop system (16) is strictly bounded real with disturbance attenuation  $g$ , then the linear matrix inequalities (LMIs) (37)-(39) have feasible solutions  $X_i, Y_i$  and  $L_i, F_i$ , where for  $i = 1, \dots, N$ , we define

$$S_i(Y) = -\text{diag}(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_N),$$

and

$$R_i(Y) = [\sqrt{\pi_{1i}}Y_i \ \dots \ \sqrt{\pi_{(i-1)i}}Y_i \ \sqrt{\pi_{(i+1)i}}Y_i \ \dots \ \sqrt{\pi_{Ni}}Y_i].$$

In this case, the controller is given by

$$C_{Ki} = F_i Y_i^{-1}, \quad (40)$$

$$B_{Ki} = (Y_i^{-1} - X_i)^{-1} L_i, \quad (41)$$

$$A_{Ki} = (Y_i^{-1} - X_i)^{-1} M_i Y_i^{-1}, \quad (42)$$

where

$$\begin{aligned} M_i &= -A_i^T - X_i A_i Y_i - X_i B_2 F_i - L_i C_2 Y_i - \\ &C_1^T (C_1 Y_i + D_{12} F_i) - g^{-2} (X_i B_1 + L_i D_{21}) B_1^T - \\ &\sum_{j=1}^N \pi_{ij} Y_j^{-1} Y_i. \end{aligned}$$

Similarly, if the LMIs (37)-(39) have feasible solutions, the closed-loop system (16) with the controller whose parameters are in (40)-(42) is strictly bounded real with the disturbance attenuation  $g$ .

**Proof.** We directly have this theorem from the corresponding classical  $H^\infty$  control results in (De Farias et al., 2000).

For any designed controller  $(A_{Ki}, B_{Ki}, C_{Ki})$  in (40)-(42), we can check the physical realisation using a similar method in (James et al., 2008). It should be noted that the theorem 5 only gives the parameters  $(A_{Ki}, B_{Ki}, C_{Ki})$ , while the parameters  $B_{\nu i}$  and  $D_{\nu i}$  are not constructed from the  $H^\infty$  control method. This gives the degree of freedom to obtain a physically realisable controller by introducing additional quantum noises and constructing the corresponding input matrices using the algorithm in (Vuglar and Petersen, 2016).

## 5. CONCLUSION

This paper illustrated the  $H^\infty$  control for a class of Markovian jump linear quantum systems, which represents a class of quantum systems suffering from fault signals. The strict bound real lemma of time-varying quantum systems is obtained, which relates the  $H^\infty$  problem to the solutions of a group of Linear Matrix Inequalities. The future work may include the  $H^\infty$  control design for Markovian jump linear quantum systems with unknown or partially known transit rate matrix, and the possible applications of  $H^\infty$  control for Markovian jump linear quantum system in quantum optics also need to be further considered.

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$$\begin{bmatrix} A_i^T X_i + X_i A_i + L_i C_2 + C_2^T L_i^T + C_1^T C_1 + \sum_{j=1}^N \pi_{ij} X_j & X_i B_1 + L_i D_{21} \\ B_1 X_i + D_{21}^T L_i^T & -g^2 I \end{bmatrix} < 0, \quad (37)$$

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} > 0, \quad (38)$$

$$\begin{bmatrix} A_i Y_i + Y_i A_i^T + B_2 F_i + F_i^T B_2^T + \pi_{ii} Y_i + g^{-2} B_1 B_1^T (C_1 Y_i + D_{12} F_i)^T & R_i(Y) \\ C_1 Y_i + D_{12} F_i & -I \\ R_i^T(Y) & 0 \\ & 0 & S_i(Y) \end{bmatrix} < 0. \quad (39)$$

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